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The effects of variable fluid properties and thermocapillarity on the flow of a thin film on an unsteady stretching sheet

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Abstract

The effects of variable viscosity, variable thermal conductivity and thermocapillarity on the flow and heat transfer in a laminar liquid film on a horizontal stretching sheet is analyzed. Using a similarity transformation the governing time dependent boundary layer equations for momentum and thermal energy are reduced to a set of coupled ordinary differential equations. The resulting five-parameter problem is solved numerically for some representative value of the parameters. It is shown that the film thickness increases with the increase in viscosity of the fluid. In other words viscosity resists film thinning. Further it is shown that more heat flows out of the liquid through the stretching surface when conductivity increases with temperature than that for the case when conductivity decreases with temperature.

Keywords: Thin film; Boundary layer; Thermocapillarity; Variable viscosity; Variable thermal conductivity

1. Introduction

Flow dynamics due to the stretching of a boundary along with heat transfer is relevant in extrusion process. In particular, in the extrusion of a polymer in a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a thin sheet, and then solidified through quenching or gradual cooling by direct contact with water or coolant liquid. Crane [1] gave an exact similarity solution in closed analytical form for steady two-dimensional boundary layer flow caused by the stretching of a flat sheet which moves in its own plane with velocity varying linearly with distance from a fixed point. Due to its practical applications, the stretching sheet problem has attracted several researchers [2–12] for the last three decades and is extensively studied to understand the same. In all these studies, boundary layer equation is considered and the boundary conditions are prescribed at the sheet and on the fluid at infinity. Imposition of similarity transformation reduces the system to a set of ODE, which is then solved either analytically or numerically. Wang [13] further widened its horizon to study the flow of liquid film on an unsteady stretching surface. Using Wang's special type of similarity transformation, Andersson et al. [14] have studied the unsteady stretching flow in case of finite thickness for power-law fluid. Later on Andersson et al. [15] have extended Wang's unsteady thin film stretching problem to the case of heat transfer.

Recently Dandapat et al. [16] studied the effect of the thermocapillarity on the hydrodynamics and heat transfer in a liquid film on a stretching surface. Thermocapillarity induces surface-tension gradients along the horizontal interface between the passive gas and the liquid film. These surface-tension gradients generate an interfacial flow that, through viscous drag, either oppose or support the shear-driven motion due to stretching sheet. Furthermore, the

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Nomenclature

A	viscosity variation parameter
b	stretching rate (s^{-1})
C_f	local skin friction coefficient, Eq. (27)
c_p	specific heat ($J\,kg^{-1}\,K^{-1}$)
f	dimensionless stream function, Eq. (14)
h	film thickness (m)
k	thermal conductivity ($W\,m^{-1}\,K^{-1}$)
M	thermocapillarity number, Eq. (25)
Ma	Marangoni number, Eq. (26)
Nu_x	local Nusselt number, Eq. (28)
Pr	Prandtl number, $Pr = \nu_0\rho_0 c_p/k_0$
q	heat flux, $k_0\partial T/\partial y$ ($J\,s^{-1}\,m^{-2}$)
Re_x	local Reynolds number, Ux/ν_0
S	unsteadyness parameter, α/b
t	time (s)
T	temperature (K)
U	sheet velocity ($m\,s^{-1}$)
u	horizontal velocity component ($m\,s^{-1}$)
v	vertical velocity component ($m\,s^{-1}$)
x	horizontal coordinate (m)
y	vertical coordinate (m)

Greek symbols

α	constant (s^{-1})
β	dimensionless film thickness
δ	thermal conductivity variation parameter
η	similarity variable, Eq. (16)
γ	constant (K^{-1})
ξ	constant (K^{-1})
θ	dimensionless temperature
μ	dynamic viscosity ($kg\,m^{-1}\,s^{-1}$)
ν	kinametic viscosity ($m^2\,s^{-1}$)
ρ	density ($kg\,m^{-3}$)
σ	surface-tension ($kg\,s^{-2}$)
τ	shear stress, $-\mu\partial u/\partial y$ ($kg\,m^{-1}\,s^{-2}$)
ψ	stream function ($m^2\,s^{-1}$)

Subscripts

i	isothermal sheet
o	origin
ref	reference value
s	sheet
x	local value

presence of thermocapillarity couples the hydrodynamic and the thermal boundary layer problems. Since there is appreciable temperature difference between the plate and the fluid and further it is well known that viscosity, surface-tension and conductivity of the fluid are strongly dependent on temperature, the variations of these physical quantities with temperature tempted us to study the effect of these variations in the flow dynamics.

It will be demonstrated that exact similarity can be achieved also in the presence of thermocapillarity, variable viscosity and thermal conductivity. Accurate numerical solutions will be provided for the resulting five-parameter problem.

2. Mathematical formulation

2.1. Governing equations and boundary conditions

Consider a thin elastic sheet that emerges from a narrow slit at the origin of a Cartesian coordinate system as depicted schematically in Fig. 1. The continuous sheet at $y = 0$ moves in its own plane with the velocity

$$U = bx/(1 - \alpha t) \tag{1}$$

where b and α are both positive constants with dimension time^{-1}. The surface temperature T_s of the stretching sheet varies with the distance x from the slit as

$$T_s = T_0 - (1/2)T_{ref} \cdot Re_x(1 - \alpha t)^{-1/2} = T_0 - T_{ref}[bx^2/2\nu_0](1 - \alpha t)^{-3/2}. \tag{2}$$

Here,

$$Re_x = Ux/\nu_0 = bx^2/\nu_0(1 - \alpha t) \tag{3}$$

is a local Reynolds number based on the sheet velocity U. T_0 is the temperature at the slit and T_{ref} can be taken as a constant reference temperature such that $0 \leqslant T_{ref} \leqslant T_0$. The expression (1) for the sheet velocity $U(x, t)$ shows that the elastic sheet, which is fixed at the origin, is stretched by applying a force in the positive x-direction. The effective stretching rate $b/(1 - \alpha t)$ increases with time since $\alpha > 0$. The expression (2) for the temperature $T_s(x, t)$ of the sheet represents a situation in which the sheet temperature decreases from T_0 at the slit in proportion to x^2 and such that the amount of temperature reduction along the sheet increases with time.

A thin liquid film of uniform thickness $h(t)$ lies on the horizontal sheet (cf. Fig. 1). The fluid motion within the film is primarily caused by the stretching of the elastic sheet. We have neglected the effect of latent heat due to evaporation by assuming the liquid to be nonvolatile. Further buoyancy is neglected due to the relatively thin liquid layer but it is not so thin that intermolecular forces come into play. Variation of the viscosity, surface-tension and thermal conductivity with temperature are assumed to be in the form:

$$\mu = \mu_0 e^{-\xi(T - T_0)}, \tag{4}$$

$$\sigma = \sigma_0[1 - \gamma(T - T_0)] \tag{5}$$

and

$$k = k_0[1 + c(T - T_0)], \tag{6}$$

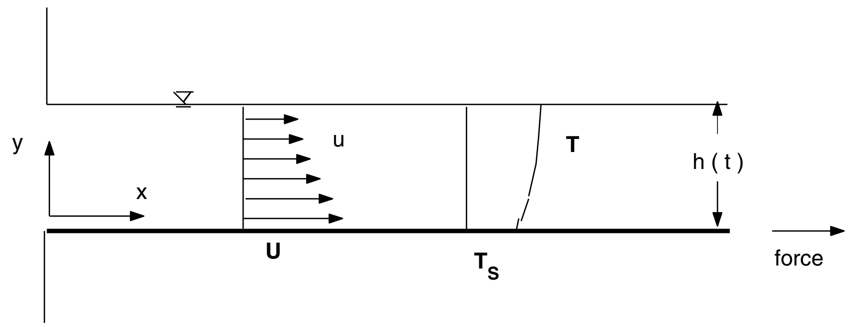

Fig. 1. Schematic diagram of the flow problem.

where μ_0, σ_0 and k_0 are the viscosity, surface-tension and conductivity of the fluid respectively at slit temperature T_0. For most liquids the surface-tension and viscosity decreases with temperature, i.e. γ and ξ are positive fluid properties. In general $c > 0$ for fluids such as water and air, while $c < 0$ for fluids such as lubrication oils. The velocity and temperature fields in the thin liquid layer are governed by the two-dimensional boundary layer equations for mass, momentum and thermal energy:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{7}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{8}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho_0 c_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right). \tag{9}$$

Here viscous dissipation of energy has been assumed negligible. The pressure is constant in the surrounding gas phase and the gravity force gives rise to a hydrostatic pressure variation in the liquid film.

In order to justify the boundary layer approximation, the length scale in the primary flow direction must be significantly larger than the length scale in the cross-stream direction. Now, if $(v_0/b)^{1/2}$ is a representative measure of the film thickness, the scale ratio $x/(v_0/b)^{1/2} \gg 1$. It is readily seen that the local Reynolds number in Eq. (3) initially equals the square of this scale ratio. Thus, just as in aerodynamic boundary layer theory, cross-stream diffusion of momentum and thermal energy can only be neglected at high Reynolds numbers.

The associated boundary conditions are

$$u=U;\quad v=0;\quad T=T_s\quad \text{at } y=0, \tag{10}$$

$$\mu\partial u/\partial y=\partial\sigma/\partial x\quad \text{at } y=h, \tag{11}$$

$$\partial T/\partial y=0\quad \text{at } y=h, \tag{12}$$

$$v=\mathrm{d}h/\mathrm{d}t\quad \text{at } y=h. \tag{13}$$

Here, it is implicitly assumed that the mathematical problem is defined only for $x \geqslant 0$. It is also assumed that the surface of the planar liquid film is smooth and free of surface waves. Although the influence of interfacial shear due to the quiescent atmosphere is negligible but a balance between the viscous shear stress $\tau = -\mu\partial u/\partial y$ and thermal stress prevails which is represented in (11). The heat flux $q = -k\partial T/\partial y$ vanishes at the adiabatic free surface, cf. Eq. (12), whereas Eq. (13) imposes a kinematic constraint on the fluid motion.

2.2. Similarity transformation

Let us introduce dimensionless variables f and θ and the similarity variable η as

$$\psi=\{v_0 b(1-\alpha t)^{-1}\}^{1/2}\cdot x\cdot f(\eta), \tag{14}$$

$$T=T_0-T_{\mathrm{ref}}[bx^2/2v_0](1-\alpha t)^{-3/2}\theta(\eta), \tag{15}$$

$$\eta=(b/v_0)^{1/2}(1-\alpha t)^{-1/2}y, \tag{16}$$

in which $\psi(x,y,t)$ is the physical stream function which automatically assures mass conservation (7). The velocity components are readily obtained as

$$u=\partial\psi/\partial y=bx(1-\alpha t)^{-1}f'(\eta), \tag{17}$$

$$v=-\partial\psi/\partial x=-\{v_0 b(1-\alpha t)^{-1}\}^{1/2}f(\eta). \tag{18}$$

The mathematical problem defined in Eqs. (7)–(13) are then transformed into a set of ordinary differential equations and their associated boundary conditions:

$$S(f'+\eta f''/2)+(f')^2-ff''=\mathrm{e}^{A\theta}[f'''+Af''\theta'], \tag{19}$$

$$Pr[(S/2)(3\theta+\eta\theta')+2\theta f'-\theta' f] = -\delta(\theta')^2+(1-\delta\theta)\theta'', \tag{20}$$

$$f'(0)=1,\ f(0)=0,\ \theta(0)=1, \tag{21}$$

$$f''(\beta)=M.\theta(\beta)\mathrm{e}^{-A\theta(\beta)}, \tag{22}$$

$$f(\beta)=S\beta/2, \tag{23}$$

$$\theta'(\beta)=0, \tag{24}$$

where $\theta=(T-T_0)/(T_s-T_0)$, $\delta=-c(T_s-T_0)$, $A=-\xi(T_s-T_0)$ and prime denotes differentiation with respect to η. It is to be noted here that A is positive as $T_0 > T_s$.

The five dimensionless parameters appear explicitly in the transformed problem. These are the unsteadiness parameters $S=\alpha/b$, the Prandtl number $Pr=v_0\rho_0 c_p/k_0$, the variable viscosity parameter A, thermal conductivity parameter δ and the thermocapillary parameter

$$M \equiv \frac{\gamma\sigma_0 T_{\rm ref}}{\mu_0 (b\nu_0)^{1/2}}. \tag{25}$$

The parameter M, which emerges naturally from the similarity analysis, is closely related to the Marangoni number, a frequently used parameter in the analysis of thermocapillary-driven flows and involves a characteristic length scale. In the present context the thickness of the liquid layer is of the order $(\nu_0/b)^{1/2}$ and the Marangoni number based on this scale becomes:

$$Ma \equiv \frac{\gamma\sigma_0 T_{\rm ref}\sqrt{\nu_0/b}\rho_0 c_p}{\mu_0 k_0} = Pr \cdot M. \tag{26}$$

In the transformed problem, boundary conditions are imposed at $\eta = 0$ and at $\eta = \beta$, the value of the similarity variable η at the free surface. Thus Eq. (16) gives $\beta = (b/\nu_0)^{1/2}(1-\alpha t)^{-1/2} h$ for $y = h$. β is a yet unknown constant that should be determined as an integral part of the boundary-value problem. The kinematic constraint (13) at $y = h(t)$ thus transforms into the free surface condition (23) and the interfacial stress balance (11) leads to (22) which serves to couple the momentum boundary layer problem to the thermal boundary layer problem. Of particular relevance is the local skin friction coefficient

$$C_{\rm f} \equiv \frac{2\tau_{\rm s}}{\rho U^2} = -2{\rm e}^A f''(0) \cdot Re_x^{-1/2} \tag{27}$$

and the local Nusselt number

$$Nu_x \equiv -\frac{x}{T_{\rm ref}}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{2}(1-\alpha t)^{-1/2} \cdot \theta'(0) \cdot Re_x^{3/2}, \tag{28}$$

where Re_x is the local Reynolds number defined in Eq. (3). $C_{\rm f}$ decreases linearly with the distance from the slit, whereas Nu_x increases as x^3.

3. Numerical procedure

The non-linear differential equations (19) and (20) subject to the boundary conditions (21)–(24) constitute a two-point boundary value problem, which was solved by the method of adjoints. The two ODEs (19) and (20) were first formulated as a set of five first-order equations. For a tentative value of β, this set subjected to the three explicit conditions (21), the explicit terminal condition (24) and the implicit terminal condition (22) was solved by the method of adjoints. The numerical solution did generally not satisfy the auxiliary terminal condition (23), and the estimated value of β was therefore systematically adjusted until Eq. (23) was satisfied to within 10^{-4}. For non-linear two-point boundary-value problems, the method of adjoints involves forward integration of the five ODEs and multiple backward integrations of the five corresponding adjoint equations, i.e. equations which are adjunct to analytically determined variational equations. The iterative process, as described in more detail in chapter 3 of Roberts and Shipman [17], was terminated when Eqs. (22) and (24) were satisfied to within 10^{-8}.

4. Results and discussion

In this study our main focus is to study the effects of viscosity and conductivity variations with temperature on hydrodynamic and thermal characteristics of the overlying fluids above the stretching surface.

4.1. Effects of viscosity variation

Fig. 2 shows the effect of viscosity variation on velocity profile $f'(\eta)$ where η varies from the stretching sheet ($\eta = 0$) to the free surface of the film ($\eta = \beta$). From this figure, it is clear that dimensionless film thickness β increases with the increase in viscosity of the fluid. Further it can be seen that the increase of viscosity increases $f'(\eta)$ near the stretching surface. It is to be noted here that the thermocapillarity produces an outward flow (for the present problem) along the free surface and it does not affect the flow near the stretching surface. As a result flow near to the stretching surface is solely guided by the action of viscous stress. Fig. 3 reveals that the local skin friction coefficient $-f''(0)$ decreases with the increase in viscosity variation parameter A. Further it is to be noted that the free surface velocity $f'(\beta)$ is decreasing with increasing A due to thickening of the liquid layer. This trend can also be observed from Figs. 2 and 4.

Fig. 5 depicts the variation of temperature profile $\theta(\eta)$ due to viscosity variation. Local heat transfer coefficient $-\theta'(0)$ at the sheet increases with the increase in viscosity as pictured in Fig. 7.

4.2. Effects of conductivity variation

Influence of conductivity variation (δ) on velocity profile is not impressive although dimensionless film thickness β changes moderately with δ as depicted in Fig. 6. But the

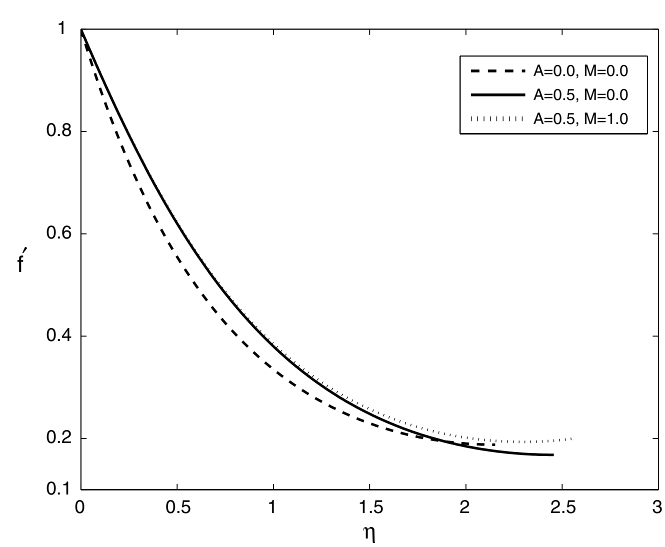

Fig. 2. Similarity velocity profiles $f'(\eta)$ for $\delta = 0.0$, $Pr = 1.0$, $S = 0.8$ and for different values of A and M.

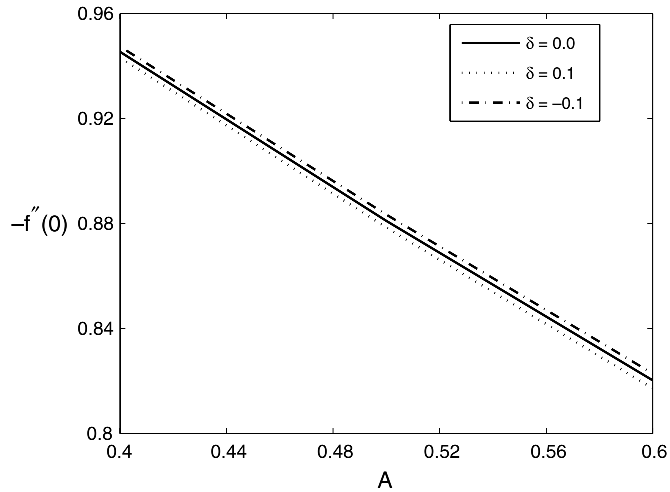

Fig. 3. Sheet shear stress $-f''(0)$ vs. viscosity variation parameter A for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

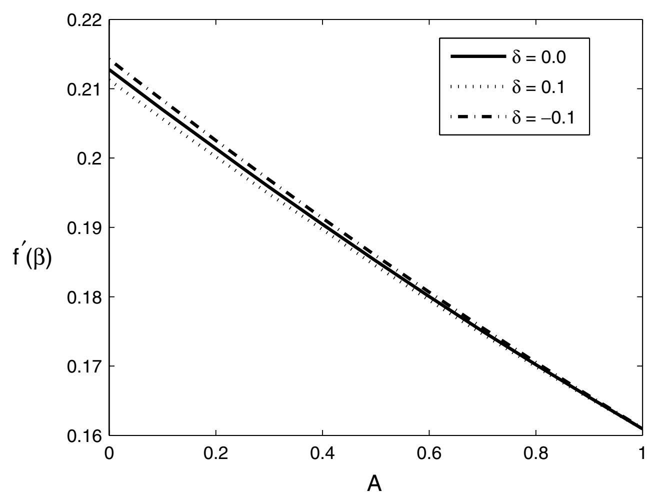

Fig. 4. Free surface velocity $f'(\beta)$ vs. viscosity variation parameter A for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

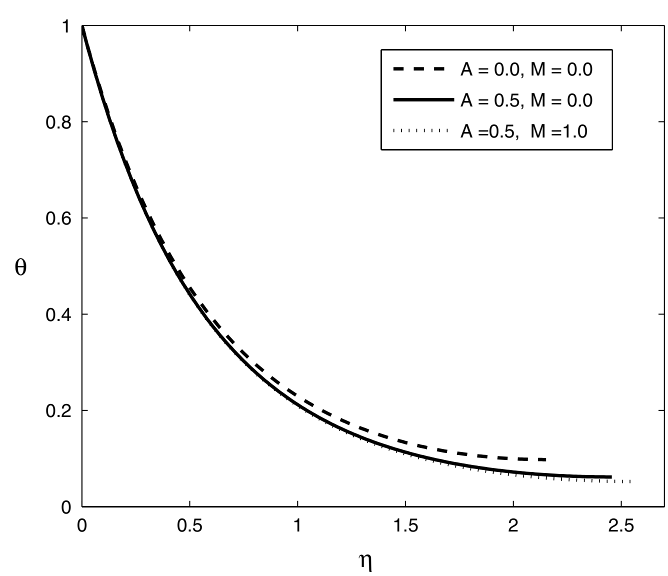

Fig. 5. Similarity temperature profiles $\theta(\eta)$ for $\delta = 0.0$, $Pr = 1.0$, $S = 0.8$ and for different values of A and M.

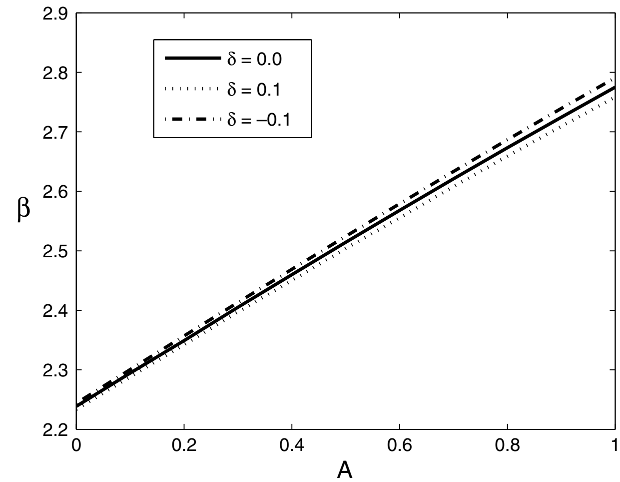

Fig. 6. Film thickness β vs. viscosity variation parameter A for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

thermal characteristics are considerably influenced that can be readily observed from Figs. 7–9. Fig. 7 displays the local rate of heat transfer $-\theta'(0)$ at the stretching sheet. The figure shows that more heat flows out of the liquid through the stretching surface when conductivity increases with temperature (represented by dotted line) than that for the case when conductivity decreases with temperature (represented by dash-dot line). As a result fluid losses more temperature when $\delta = 0.1$ than that for the corresponding case when $\delta = -0.1$. This phenomena are depicted in Figs. 8 and 9. Higher heat transfer rate ($\delta = 0.1$) at the stretching sheet allows the adjacent liquid to cool faster than that for the case ($\delta = -0.1$) of lower heat transfer rate. As a result local skin friction coefficient $-f''(0)$ is less for the case when $\delta = 0.1$ in comparison with that for $\delta = -0.1$ and it can be observed from Fig. 3.

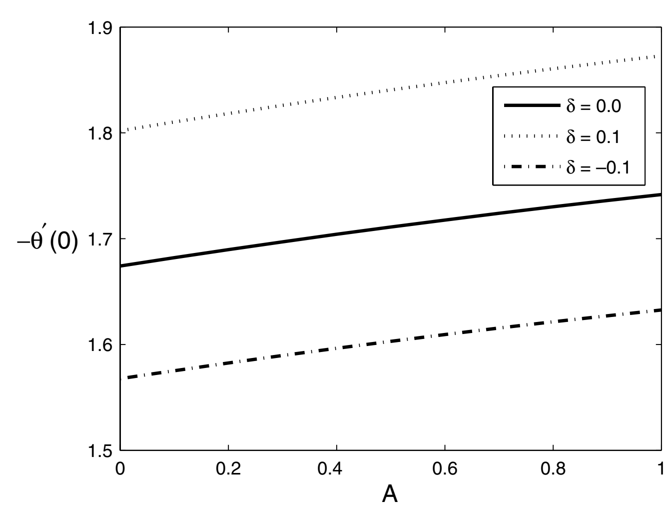

Fig. 7. Dimensionless heat flux $-\theta'(0)$ vs. viscosity variation parameter A for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

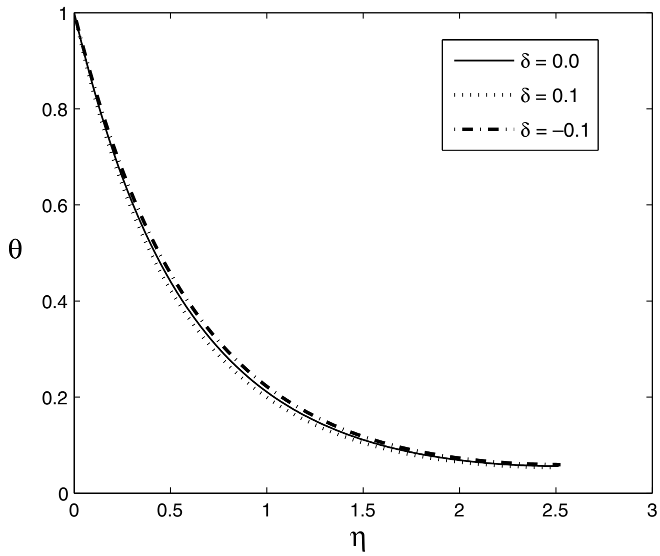

Fig. 8. Similarity temperature profiles $\theta(\eta)$ for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

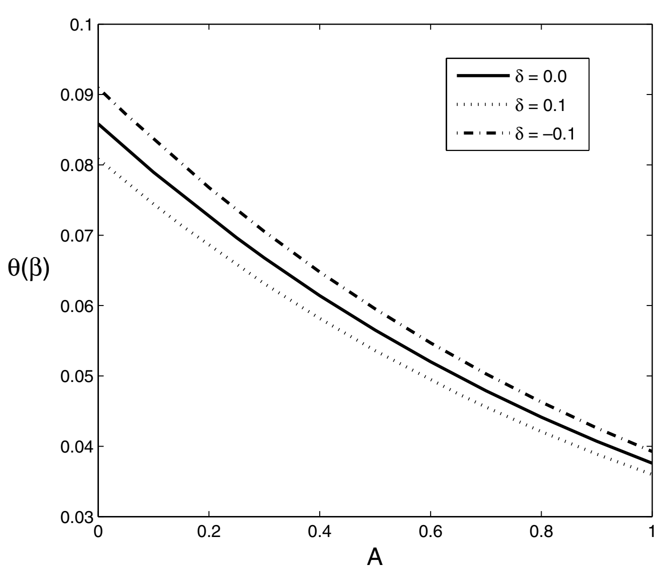

Fig. 9. Dimensionless surface temperature $\theta(\beta)$ vs. viscosity variation parameter A for $M = 0.5$, $Pr = 1.0$, $S = 0.8$ and for different values of δ.

5. Conclusion

Unsteady flow of a thin liquid film over a stretching sheet is studied in the light of variation of fluid properties due to temperature differences. In this study emphasis is given on how velocity field, skin friction, temperature distribution and heat transfer changes due to the variation of viscosity, conductivity and thermocapillarity with temperature. We have used a similarity transformation that reduces the governing equations to a system of non-linear ODEs which are then solved numerically by the shooting method (method of adjoint).

It is found that velocity field increases when viscosity increases near the sheet. In other words, increase in viscosity produces increased shear stress which results velocity increase near the sheet. However variation of conductivity with temperature do not influence the velocity profiles impressively although film thickness changes appreciably. Further increase in conductivity with temperature increases heat flow through the stretching surface.

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